Journal of Biostatistics and Biometric Applications
Volume 3 | Issue 3
ISSN: 2455-765X

Review Article **Open Access**

A Profile of Trigonometric Measures of Fuzzy Information and Discrimination

Hooda DS*

Former PVC, Kurukshetra University, Honorary Professor in Mathematics, GJ university of Science and Technology, Hisar-125001, India

***Corresponding author:** Hooda DS, Former PVC, Kurukshetra University, Honorary Professor in Mathematics, GJ university of Science and Technology, Hisar-125001, India, E-mail: ds_hooda@rediffmail.com

Citation: Hooda DS (2018) A Survey on Trigonometric Measures of Fuzzy Information and Discrimination. J Biostat Biometric App 3(3): 303

Received Date: August 7, 2018 **Accepted Date**: September 3, 2018 **Published Date:** September 6, 2018

Abstract

In the literature of fuzzy information measures, there exist many well-known parametric and non-parametric measures with their own merits and demerits. One important category of these measures is transcendental measures consisting of logarithmic, exponential and trigonometric fuzzy information and discrimination. These measures have wide applications in pattern recognition, medical diagnosis and MCDM problems. However, trigonometric fuzzy information measures have their own importance for application point of view particularly to geometry.

In present communication the concept of fuzzy information measure is introduced with a few generalizations. A brief survey of trigonometric measures of fuzzy information and discrimination characterized by various authors is described. The Application of one generalized measure of fuzzy discrimination in strategic decision making is explained and illustrated with an example. Conclusion with an exhaustive list of references is also given in the end.

Keywords: Trigonometric Measures; Fuzzy Information

Introduction

The concept of entropy was introduced and developed to measure the uncertainty of a probability distribution by Shannon [1]. In 1965, Zadeh introduced the concept of fuzzy set and that was extended to measure uncertainty in ambiguous or imprecision linguistic statement [2]. A fuzzy set on a universe of discourse $X = \{x_1, x_2, ..., x_n\}$ was defined by Zadeh as given below [2]:

$$A = \left\{ \left\langle x, \mu_A(x) \right\rangle / x \in X \right\}$$

where is the membership function of . The membership value describes the degree of the belongingness of in . The closer the value of $\mu_A(X)$ is to 1, the more x belongs to A.

The specificity of fuzzy sets is to capture the idea of partial membership. The characteristic function of a fuzzy set is often called membership function and the role of that has well been explained by Singpurwalla and Booker [3] in probability measures of fuzzy sets.

Fuzzy set theory, in one or other way, is wide applications in many areas of science and technology e.g. clustering, image processing, decision making, pattern recognition, medical diagnosis and multi-criteria decision making, because of its capability to model non-statistical imprecision or vague concepts. When proposing fuzzy set, Zadeh's [2] concerns were explicitly centred on their potential contribution in the domain of pattern classification, processing and communication of information, abstraction, summarization, etc.

A generalized theory of uncertainty was well explained by Zadeh [4] where he remarked that uncertainty was an attribute of information. Before that the path breaking work of Shannon had led to a universal acceptance of the theory that information was statistical in nature [1]. However, a perception-based theory of probabilistic reasoning of imprecise or vague concepts was explained by Zadeh [5].

Some work related to fuzzy uncertainty management for intelligence analysis was reported by Yager, [6] whereas the generalized fuzzy information theory, its aims, results and some open problems were discussed by Klir [7]. Nanda and Paul redefined Khichin's

[8] version of entropy and discussed the properties of the aging classes based on their generalized entropy.

Taking into consideration the concept of fuzzy set, De Luca and Termini [9] suggested that corresponding to Shannon's [1] probabilistic entropy, the measure of fuzzy information could be defined as follows:

$$H(A) = -\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right], \tag{1.1}$$

where $\mu_A(x_i)$are the membership values?

Fuzzy entropy is a measure of fuzziness of a set which arises from the intrinsic ambiguity or vagueness possessed in the fuzzy set. Bhandari and Pal [10] defined logarithmic fuzzy information, which is a generalization of (1.1) and is given below [10]:

$$H_\alpha(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right];\ \alpha > 0,\ \alpha \neq 1. \tag{1.2}$$

On the same lines many researchers have studied various generalized fuzzy information measures. Hooda, [11] Hooda and Bajaj [12] and many other authors introduced and characterized various generalized additive and non-additive fuzzy information measures and their applications. Mishra *et al.* [13] proposed exponential fuzzy measures

Hooda and Jain [14] characterized a sub additive trigonometric measure of fuzzy information of type α and degree β corresponding to trigonometric probabilistic entropy studied by Sharma and Taneja [15]. It may be noted that trigonometric measures have their own importance in application point of view, particularly, in geometry.

Monte *et al.* [16] introduced an axiomatic definition of divergence measure for fuzzy sets and discussed its several properties. They pointed out that divergence measures were described to compute the discrimination among fuzzy sets that was why these were named as fuzzy measures of discrimination. Interestingly, divergence measure is also a measure of dissimilarity which possesses a set of important properties and was applied in evaluation of discrimination measure for fuzzy sets.

In the present communication a survey of trigonometric fuzzy information measures has been made. In section 2, two sine and cosine trigonometric fuzzy information measures are described, while the characterization of a new cosine fuzzy information measure is explained in section 3. In section 4, a new tangent inverse trigonometric fuzzy information measures is described along with a generalized measure of fuzzy discrimination in section 5. In section 6, application of fuzzy discrimination measure in strategic decision making is explained and illustrated with numerical example.

Sine and Cosine Trigonometric Fuzzy Information Measures

Most of fuzzy information measures have been defined analogous to probabilistic entropies. Hooda and Mishra [17] introduced two trigonometric fuzzy information measures which have no analogous probabilistic entropies as given below:

$$\text{(1)}\ H_1(A) = \sum_{i=1}^{n}\left[\sin\frac{\pi\mu_A(x_i)}{2} + \sin\frac{\pi(1-\mu_A(x_i))}{2} - 1\right] \tag{2.1}$$

$$\text{(2)}\ H_2(A) = \sum_{i=1}^{n}\left[\cos\frac{\pi\mu_A(x_i)}{2} + \cos\frac{\pi(1-\mu_A(x_i))}{2} - 1\right] \tag{2.2}$$

First of all we check the validity of the proposed measures (2.1) and (2.2).

Theorem 2.1[9]: The fuzzy information measure given by (2.1) and (2.2) are valid measures.

Proof: To prove that the given measure is a valid measure, we shall show that (2.1) satisfies the four properties (P1) to (P4).

(P1). H_1 (A)=0 if and only if A is a crisp set.

Evidently, $H_1(A) = \sum_{i=1}^{n}\left[\sin\frac{\pi\mu_A(x_i)}{2} + \sin\frac{\pi(1-\mu_A(x_i))}{2} - 1\right] = 0$ if and only if either $\mu_A(x_i) = 0$ or $1-\mu_A(x_i) = 0$ for $i = 1,2,\ldots\ldots,n.$

It implies H_1 (A)=0 if and only if A is a crisp set.

(P2). H_1 (A) is maximum if and only if A is the fuzziest set *i.e.,* $\mu_A(x_i) = 0.5$ for all $i = 1,2,\ldots\ldots,n.$

Differentiating H_1 (A) with respect to $\mu_A(x_i)$, we have

$$\frac{dH_1(A)}{d\mu_A(x_i)} = \frac{\pi}{2}\sum_{i=1}^{n}\left[\cos\frac{\pi\mu_A(x_i)}{2} - \cos\frac{\pi(1-\mu_A(x_i))}{2}\right], \tag{2.3}$$

which vanishes at $\mu_A(x_i)=0.5$.

Again differentiating (2.3) with respect to $\mu_A(x_i)$, we get

$$\frac{d^2H_1(A)}{d\mu_A(x_i)^2} = \frac{\pi^2}{4}\sum_{i=1}^{n}\left[-\sin\frac{\pi\mu_A(x_i)}{2} - \sin\frac{\pi(1-\mu_A(x_i))}{2}\right], \tag{2.4}$$

which is less than zero (< 0) at $\mu_A(x_i)=0.5$.

Hence H_1 (A) is maximum at $\mu_A(x_i)=0.5$ for $i=1,2,\ldots\ldots,n.$

Further from (2.3) we see that $H_1(A)$ is an increasing function of $\mu_A(x_i)$ in the region $0 \le \mu_A(x_i) \le 0.5$ and $H_1(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0.5 \le \mu_A(x_i) \le 1$.

(P3). Let A^* be sharpened version of A, which means that

if $0 \le \mu_A(x_i) \le 0.5, \mu_{A^*}(x_i) \le \mu_A(x_i)$ for all $i=1,2,\ldots\ldots,n.$

and if $0.5 \le \mu_A(x_i) \le 1, \mu_{A^*}(x_i) \ge \mu_A(x_i)$ for all $i=1,2,\ldots\ldots,n.$.

Since $H_1(A)$ is an increasing function of $\mu_A(x_i)$ in the region $0 \le \mu_A(x_i) \le 0.5$ and $H_1(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0 \le \mu_A(x_i) \le 0.5$, therefore

(i) $\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow H_1(A^*) \le H_1(A)$ in $[0,0.5]$ (2.5)

(ii) $\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow H_1(A^*) \le H_1(A)$ in $[0.5,1]$. (2.6)

Thus, (2.5) and (2.6) together give

$$H_1(A^*) \le H_1(A)$$

(P4). From the definition it is evident that

$$H_1(A) = H_1(A^c)$$

where A^c is complement of A obtained by replacing $\mu_A(x_i)$ by $1-\mu_A(x_i)$.

Hence H_1 (A) satisfies all the essential four properties of fuzzy information measures. Thus it is a valid measure of fuzzy information. On the same line it can be proved that H_2 (A) is a valid fuzzy information measure.

By considering a concave function $\sin \pi x, \forall x \in [0,1]$ Hooda and Mishra [17] defined the following fuzzy information measure and proved its validity:

$$H_3(A) = \sum_{i=1}^{n}\left[\sin \pi\mu_A(x_i) + \sin \pi(1-\mu_A(x_i))\right] \tag{2.7}$$

Theorem 2.2[8]: The fuzzy information measure given by (2.7) is valid measure.

Proof: To prove that the given measure is a valid measure, we shall show that (2.7) satisfies the four properties (P1) to (P4).

(P1). $H_3(A)$=0 if and only if A is a crisp set.

Evidently, $H_3(A)=\sum_{i=1}^{n}\left[\sin\pi\mu_A(x_i)+\sin\pi(1-\mu_A(x_i))\right]=0$ if and only if either $\mu_A(x_i)=0$ or $1-\mu_A(x_i)=0$ for $i=1,2,\ldots\ldots,n.$

It implies $H_3(A)=0$ if and only if A is a crisp set.

(P2). $H_3(A)$ is maximum if and only if A is the fuzziest set i.e.,$\mu_A(x_i)=0.5$ for all $i=1,2,\ldots\ldots,n.$

Differentiating $H_3(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{dH_3(A)}{d\mu_A(x_i)}=\pi\sum_{i=1}^{n}\left[\cos\pi\mu_A(x_i)-\cos\pi(1-\mu_A(x_i))\right] \tag{2.8}$$

which vanishes at $\mu_A(x_i)=0.5$.

Again differentiating (2.8) with respect to $\mu_A(x_i)$, we get

$$\frac{d^2H_3(A)}{d\mu_A(x_i)^2}=-\pi^2\sum_{i=1}^{n}\left[\sin\pi\mu_A(x_i)+\sin\pi(1-\mu_A(x_i))\right],$$

which is less than zero (<0) at $\mu_A(x_i)=0.5$.

Hence $H_3(A)$ is maximum at $\mu_A(x_i)=0.5$ for $i=1,2,\ldots\ldots,n.$

$H_3(A)$ is an increasing function of $\mu_A(x_i)$ in the region $0\le\mu_A(x_i)\le0.5$ and $H_3(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0.5\le\mu_A(x_i)\le1$.

(P3). Let A^* be sharpened version of A, which means that

if $0\le\mu_A(x_i)<0.5,\mu_{A^*}(x_i)\le\mu_A(x_i)$ for all $i=1,2,\ldots\ldots,n.$

And if $0.5<\mu_A(x_i)\le1,\mu_{A^*}(x_i)\ge\mu_A(x_i)$ for all $i=1,2,\ldots\ldots,n.$.

Since $H_3(A)$ is an increasing function of $\mu_A(x_i)$ in the region $0\le\mu_A(x_i)\le0.5$ and $H_3(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0.5\le\mu_A(x_i)\le1$.

Therefore,

(i) $\mu_{A^*}(x_i)\le\mu_A(x_i)\Rightarrow H_3(A^*)\le H_3(A)$ in $[0,0.5]$ (2.9)

$(ii)\,\mu_{A^*}(x_i)\ge\mu_A(x_i)\Rightarrow H_3(A^*)\le H_3(A)$ in $[0.5,1]$. (2.10)

Hence (2.9) and (2.10) together give

$$H_3(A^*)\le H_3(A)$$

(P4). It is evident from the definition that

$$H_3(A)=H_3(A^c)$$

where A^c is complement of A obtained by replacing $\mu_A(x_i)$ by $1-\mu_A(x_i)$.

Hence $H_3(A)$ satisfies all the essential four properties of fuzzy information measures. Thus it is a valid measure of fuzzy information.

Further they defined generalized sine trigonometric fuzzy information measure as given below:

$$H_4(A)=\sum_{i=1}^{n}\left[\sin\beta\mu_A(x_i)+\sin\beta(1-\mu_A(x_i))-\sin\beta\right]. \tag{2.11}$$

It may be noted that (2.11) reduces to (2.7) when $\beta=\pi$ and reduces to (2.1) , when $\beta=\frac{\pi}{2}$.

Another generalized sine trigonometric measure of fuzzy information is

$$H_5(A)=\sum_{i=1}^{n}\left[\sin\left(\beta\mu_A(x_i)+\alpha\right)+\sin\left(\beta(1-\mu_A(x_i))+\alpha\right)-\sin(\alpha+\beta)\right]. \tag{2.12}$$

In particular when $\alpha=0,$ (2.12) reduces to (2.11) and reduces to (2.7) when $\alpha=0$ and $\beta=\pi.$

A generalized cosine trigonometric measure of fuzzy information is defined as

$$H_6(A)=\sum_{i=1}^{n}\left[\cos\beta\mu_A(x_i)+\cos\beta(1-\mu_A(x_i))-(1+\cos\beta)\right], \tag{2.13}$$

which is a new generalized cosine trigonometric measure of fuzzy information and it reduces to the following fuzzy information measure when $\beta=\frac{\pi}{2}$

$$H_7(A)=\sum_{i=1}^{n}\left[\cos\frac{\pi\mu_A(x_i)}{2}+\cos\frac{\pi(1-\mu_A(x_i))}{2}-1\right]=H_2(A). \tag{2.14}$$

It can be easily verified that these fuzzy information measure satisfy the essential properties of validity.

A New Cosine Fuzzy Information Measure

Let $X=\{x_1,x_2,\ldots\ldots,x_n\}$ be a set of universe and A be a fuzzy set in X having membership function $\mu_A(x_i)$ defined on X. Then cosine fuzzy information measure for fuzzy set A was defined as

$$H_8(A)=\frac{1}{n}\sum_{i=1}^{n}\left[\cos\left(\frac{|2\mu_A(x_i)-1|}{2}\right)\pi\right] \tag{3.1}$$

Theorem 3.1 [11]: $H_8(A)$ is a valid fuzzy information measure.

Proof: For validity of the measure we shall prove that the following four essential properties (P1) to (P4) are satisfied:

(P1). $H_8(A) = 0$ if and only if A is a crisp set.

Evidently $H_8(A) = 0$ if and only if either $\mu_A(x_i) = 0$ or $1-\mu_A(x_i)=0$ for $i=1,2,\ldots\ldots,n.$

(P2). $H_8(A)$ is maximum iff $\mu_A(x_i)=0.5$ for all $i=1,2,\ldots\ldots,n.$

Differentiating $H_8(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{dH_8(A)}{d\mu_A(x_i)}=-\frac{\pi}{n}\sum_{i=1}^{n}\left[\sin\left(\frac{|2\mu_A(x_i)-1|}{2}\right)\pi\right], \tag{3.2}$$

which reduces to zero when $_A(\ _i)\ \ 0.5$ for all $i=1,2,\ldots\ldots,n.$

Differentiating again, we get

$$\frac{d^2H_8(A)}{d\mu_A(x_i)^2}=-\frac{\pi^2}{n}\sum_{i=1}^{n}\left[\cos\left(\frac{|2\mu_A(x_i)-1|}{2}\right)\pi\right],$$

which is less than zero when $\mu_A(x_i)=0.5$ for all i.

Hence $H_8(A)$ is maximum when $\mu_A(x_i)=0.5$ for all $i=1,2,\ldots\ldots,n.$ or A is the fuzziest set.

Further it may be noted that $H_8(A)$ is an increasing function of $\mu_A(x_i)$ in the region $0\le\mu_A(x_i)\le 0.5$ and $H_8(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0.5\le\mu_A(x_i)\le 1$.

(P3). Let A^* be sharpened version of A, then

(i) $\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow H_8(A^*) \le H_8(A)$ in $[0,0.5]$ (3.3)

(ii) $\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow H_8(A^*) \le H_8(A)$ in $[0.5,1]$. (3.4)

Hence from (3.3) and (3.4) we can conclude that

$$H_8(A^*) \le H_8(A)$$

(P4). It is evident that if $\mu_A(x_i)$ replacing by $1-\mu_A(x_i)$ for all $i = 1,2,\ldots\ldots,n.$, then $H_8(A^c) = H_8(A)$, where A^c is complement of A.

Theorem 3.2 [11]: Let $X = \{x_1, x_2, \ldots\ldots, x_n\}$ be the universe of discourse. Let A and B be two fuzzy set in a fixed universe of discourse X. Let $A(x_i) = \mu_A(x_i)$ and $B(x_i) = \mu_B(x_i)$ satisfying either $A \subseteq B$ or $B \subseteq A$, then the following holds:

$$H_8(A \cup B) + H_8(A \cup B) = H_8(A) + H_8(B) \tag{3.5}$$

Proof: Let us separate X into two parts X_1 and X_2, where

$$X_1 = \{\mathrm{x}_1 \in \mathrm{X} : \mathrm{A}(x) \subseteq \mathrm{B}(x)\}$$

and

$$X_2 = \{\mathrm{x}_2 \in \mathrm{X} : \mathrm{B}(x) \subseteq \mathrm{A}(x)\}.$$

It implies that for all $x_i \in X_1$, $\mu_A(x_i) \le \mu_B(x_i)$ and for all $x_i \in X_2$, $\mu_A(x_i) \ge \mu_B(x_i)$.

From (3.1), we have

$$H_8(A \bigcup B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\left(\frac{|2\mu_{A\bigcup B}(x_i) - 1|}{2}\right)\pi\right]$$

$$= \frac{1}{n}\left[\sum_{x_i \in X_1}\cos\left(\frac{|2\mu_B(x_i) - 1|}{2}\right)\pi + \sum_{x_i \in X_2}\cos\left(\frac{|2\mu_A(x_i) - 1|}{2}\right)\pi\right]. \tag{3.6}$$

Again from definition (3.1), we have

$$H_8(A \bigcap B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\left(\frac{|2\mu_{A\bigcap B}(x_i) - 1|}{2}\right)\pi\right]$$

$$= \frac{1}{n}\left[\sum_{x_i \in X_1}\cos\left(\frac{|2\mu_A(x_i) - 1|}{2}\right)\pi + \sum_{x_i \in X_2}\cos\left(\frac{|2\mu_B(x_i) - 1|}{2}\right)\pi\right]. \tag{3.7}$$

Adding (3.6) and (3.7), we get

$H_8(A \cup B) + H_8(A \cap B) = H_8(A) + H_8(B)$. Hence this theorem is proved.

Corollary 1: Let A be a fuzzy set on X and be its complement, then

$$H_8(A) = H_8(A^c) \tag{3.8}$$

and

$$H_8\left(A \cup A^c\right) = H_8\left(A \cap A^c\right) \tag{3.9}$$

Tangent Inverse Trigonometric Fuzzy Information Measure

Prakash and Gandhi [9] proposed the following tangent fuzzy information measure:

$$H_9(A) = -\sum_{i=1}^{n}\left[\tan\frac{\pi}{2n\mu_A(x_i)} + \tan\frac{\pi}{2n(1-\mu_A(x_i))} - n^2\tan\frac{\pi}{2n}\right], n > 3. \tag{4.1}$$

They proved that (4.1) was a valid fuzzy measure and applied to the geometry to estimate the perimeter and area of polygon.

Note: Hooda and Mishra also defined tangent inverse trigonometric fuzzy information measure and prove its validity [17].

Definition 4.1 Let $X = \{x_1, x_2, \ldots\ldots, x_n\}$ be a set of universe and A be a fuzzy set in having membership function $\mu_A(x_i)$ defined on X. Then tangent inverse trigonometric fuzzy information measure for fuzzy set A is defined as

$$H_\alpha(A) = \frac{2}{1-\alpha}\sum_{i=1}^{n}\left[\tan^{-1}\left\{\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right\} - \frac{\pi}{4}\right]; \quad \alpha > 0, \alpha \neq 1 \tag{4.2}$$

Theorem 4.1 [13]: The measure in (4.2) is a valid fuzzy information measure.

Proof: To prove that the measure (4.2) is a valid measure, we shall show that it is satisfying the four properties (P1) to (P4).

(P1). $H_\alpha\left(A\right) = 0$ if and only if A is a crisp set.

If $\mu_A\left(x_i\right) = 0$ or $\mu_A\left(x_i\right) = 1$, $H_\alpha\left(A\right) = 0$.

Hence $H_\alpha\left(A\right) = 0$ if A is crisp set, $i.e. \mu_A\left(x_i\right) = 0$ or $\mu_A\left(x_i\right) = 1$ for $i = 1, 2, \ldots\ldots, n.$

Conversely, $H_\alpha\left(A\right) = 0$, then

$$\frac{2}{1-\alpha}\sum_{i=1}^{n}\left[\tan^{-1}\left\{\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right\} - \frac{\pi}{4}\right] = 0$$

$$\text{or} \qquad \tan^{-1}\left\{\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right\} - \frac{\pi}{4} = 0$$

$$\text{or} \qquad \mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha = 1.$$

The equality holds if either $\mu_A\left(x_i\right) = 0$ or $\mu_A\left(x_i\right) = 1$ i.e. A is a crisp set.

Thus $H_\alpha\left(A\right) = 0$ if and only if A is a crisp set, $i.e. \mu_A\left(x_i\right) = 0$ or $\mu_A\left(x_i\right) = 1$ for $i = 1, 2, \ldots\ldots, n.$

(P2). $H_\alpha\left(A\right)$ is maximum if and only if A is the fuzziest set, $i.e. \mu_A\left(x_i\right) = 0.5$ for $i = 1, 2, \ldots\ldots, n.$

Differentiating (4.2) with respect to $\mu_A\left(x_i\right)$, we have

$$\frac{dH_\alpha(A)}{d\mu_A(x_i)} = \frac{2}{1-\alpha}\sum_{i=1}^{n}\left[\frac{\alpha\mu_A^{\alpha-1}(x_i) - \alpha(1-\mu_A(x_i))^{\alpha-1}}{1+\left\{\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right\}^2}\right] \tag{4.3}$$

Again differentiating (4.3) with respect to $\mu_A\left(x_i\right)$, we have

$$\frac{d^2H_\alpha(A)}{d\mu_A(x_i)^2}=\frac{2\alpha}{1-\alpha}\sum_{i=1}^{n}\left[\frac{\begin{array}{l}\left[1+\left\{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right\}^2\right]\left[(\alpha-1)\left\{\mu_A^{\alpha-2}(x_i)+(1-\mu_A(x_i))^{\alpha-2}\right\}\right]\\ -\left\{\mu_A^{\alpha-1}(x_i)-(1-\mu_A(x_i))^{\alpha-1}\right\}\left[2\left\{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right\}\right]\\ \times\left\{\alpha\mu_A^{\alpha-1}(x_i)-\alpha(1-\mu_A(x_i))^{\alpha-1}\right\}\end{array}}{\left[1+\left\{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\right\}^2\right]^2}\right]$$

For maxima, $\frac{dH_\alpha(A)}{d\mu_A(x_i)}=0$

It implies $\mu_A^{\alpha-1}(x_i)-(1-\mu_A(x_i))^{\alpha-1}=0$

$$\text{or}\quad \mu_A(x_i)-(1-\mu_A(x_i))=0$$

$$\text{or}\quad \mu_A(x_i)=0.5$$

Here two cases arise:

Case1. When $0<\alpha<1$, we have $\frac{d^2H_\alpha(A)}{d\mu_A(x_i)^2}<0$ (-ve) (4.4)

Case2. When $\alpha>1$, we have $\frac{d^2H_\alpha(A)}{d\mu_A(x_i)^2}<0$ (-ve) (4.5)

From (4.4) and (4.5), we have $\frac{d^2H_\alpha(A)}{d\mu_A(x_i)^2}<0$ (-ve).

Hence $H_\alpha(A)$ is maximum at $\mu_A(x_i)=0.5$ for all $i=1,2,\ldots\ldots,n$

(P3). From (4.3), we have $\frac{dH_\alpha(A)}{d\mu_A(x_i)}>0$ (+ve) in the region $0\le\mu_A(x_i)\le0.5$

Hence $H_\alpha(A)$ is an increasing function in the region $0\le\mu_A(x_i)\le0.5$

Similarly, from (4.3) $\frac{dH_\alpha(A)}{d\mu_A(x_i)}>0$ (-ve) in the region $0\le\mu_A(x_i)\le0.5$

Hence $H_\alpha(A)$ is decreasing function in the region $0\le\mu_A(x_i)\le0.5$

Let A^* be the sharpend version of A, then

(i) $\mu_{A^*}(x_i)\le\mu_A(x_i)\Rightarrow H_\alpha(A^*)\le H_\alpha(A)$ in $[0,0.5]$ (4.6)

(ii) $\mu_{A^*}(x_i)\ge\mu_A(x_i)\Rightarrow H_\alpha(A^*)\le H_\alpha(A)$ in $[0.5,1]$. (4.7)

Thus from (4.6) and (4.7) together gives

$$H_\alpha(A^*)\le H_\alpha(A)$$

(P4). Let A^c be a complement of A.

Therefore, from the definition, we have

$$H_\alpha(A)=H_\alpha(A^c)$$

Since $H_\alpha(A)$ satisfies all the properties of fuzzy information measure, therefore, it is a valid measure of fuzzy information.

Particular Case: When $\alpha\to1$, (4.2) reduces to De Luca and Termini fuzzy information measure [9].

A Generalized Measure of Fuzzy Discrimination

Analogous to Bhatia and Singh, [19] Bhandari *et al.* [20] suggested the simplest measure of fuzzy discrimination as follows:

$$I_{BP}(A,B)=\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right] \quad (5.1)$$

Analogous to (5.1), Hooda and Mishra [17] defined a new inverse tangent fuzzy discrimination measure as follows:

Definition5.1. The fuzzy discrimination measure is said to be valid if it satisfies

(i) Non negativity $i.e.\ I(A,B)\ge 0$,

(ii) $I(A,B)=0,$ if $A=B$,

(iii) $I(A,B)\ge 0$ is convex function in $]0,1[$,

(iv) $I(A,B)\ge 0$ should not change, when $\mu_A(x_i)$ is changed to $1-\mu_A(x_i)$ and $\mu_B(x_i)$ is changed to $1-\mu_B(x_i)$

Theorem 5.1 [17]: Let $X=\{x_1, x_2, ..., x_n\}$, $A,\ B\in FS(X),$ then

$$I_\alpha(A,B)=\frac{2}{\alpha-1}\sum_{i=1}^{n}\left[\tan^{-1}\left\{\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\right\}-\frac{\pi}{4}\right] \quad (5.2)$$

where $\alpha>0,\ \alpha\ne 1,$ is a valid fuzzy discrimination measure

Proof: It can be easily verified that $I_\alpha(A,B)$ satisfy all four properties from (i) to (IV). Thus it is a valid generalized measure of fuzzy discrimination.

Application of Fuzzy Discrimination in Strategic Decision Making

In current scenario, the applications of the fuzzy discrimination measure in various fields have been studied, namely, in bio-informatics by Poletti *et al.* [21] and Fan *et al.* [22].

Bhatia and Singh [19] also discussed in image thresholding, while Ghosh *et al.* [23] applied it in automated leukocyte recognition. In this section, the application of the above fuzzy discrimination measure in strategic decision making is proposed and studied.

Decision making problem is the process of finding the optimal solution from all the existing feasible alternatives. It is assumed that a firm Y desires to apply strategies to meet its goal. Let each strategy $\{S_1, S_2, ..., S_m\}$ has different degree of effectiveness. If it has different input associated with it, then let it be $\{l_1, l_2, ..., l_n\}$. The fuzzy set Y denotes the effectiveness of a particular strategy with uniform input. Then,

$$Y=\{(S,\ \mu_S(S_i))\ :i=1,\ 2,\ ......,\ m\} \quad (6.1)$$

Further, let denotes the degree of effectiveness of a strategy when it is implemented with input l_j, then

$$I=\{(Y,\ \mu_I(S_i)):\ i=1,\ 2,\ ......,\ m\}\quad j=1,\ 2,\ ......,\ n \quad (6.2)$$

Substituting $A=Y$ and $B=I$ in the fuzzy discrimination measure given by (5.2), we calculate $I_\alpha(Y,I)$. Then, the most effective is determined by

$$I_t=\min_{\substack{1\le j\le n\\ 0<\alpha<1}}\{I_\alpha(Y,\ I)\} \quad (6.3)$$

It is assumed that $I_t(1\le t\le n)$ determines the minimum value of $\{I_\alpha(Y,I)\}_{0<\alpha<1}$. With this I_t we find $\max_{1\le i\le m}\{\mu_{I_t}(S_i)\}$, let it correspond to $S_p(1\le p\le m)$. Hence, if the strategy S_p is implemented with input budget of I_t the firm will meet its goal in the most input effective manner.

Numerical Illustration

Let $m = n = 5$ in the above model. Table 1 shows the efficiency of different strategies at uniform inputs. Table 2 illustrates the efficiency of different strategies at particular inputs and Table 3 the numerical values of discrimination measure $\{I_\alpha(Y,I)\}_{\substack{1\le j\le n\\ 0<\alpha<1}}$.

Table1: Efficiency of different strategies at uniform inputs

$\mu_Y(S_1)$	$\mu_Y(S_2)$	$\mu_Y(S_3)$	$\mu_Y(S_4)$	$\mu_Y(S_5)$
0.3	0.5	0.4	0.6	0.2

Table2: Efficiency of different strategies at particular inputs

	$\mu_I(S_1)$	$\mu_I(S_2)$	$\mu_I(S_3)$	$\mu_I(S_4)$	$\mu_I(S_5)$
I_1	0.3	0.6	0.4	**0.7**	0.2
I_2	0.5	0.3	0.8	0.4	0.7
I_3	0.6	0.7	0.6	0.9	0.4
I_4	0.5	0.6	0.3	0.8	0.9
I_5	0.4	0.5	0.4	0.2	0.6

Table3: Numerical values of discrimination measure $\{I_\alpha(Y,I)\}_{\substack{1\le j\le n\\ 0<\alpha<1}}$

	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$
I_1	**0.0041885**	**0.0126504**	**0.0151538**	**0.0128018**	**0.0055147**
I_2	0.1180805	0.3591652	0.4293109	0.3587822	0.1515919
I_3	0.0694641	0.2131733	0.2591029	0.2216869	0.0964639
I_4	0.1421777	0.4548828	0.5662826	0.4871129	0.2079755
I_5	0.0745879	0.2263425	0.2704184	0.2262243	0.0957976

The calculated numerical data of the proposed fuzzy discrimination measure indicates that input I_1 is more suitable. The assessment of the results existing in Table 2 and Table 3 points out that strategy S_4 is most effectual. Thus a firm will achieve its goal most efficiently if the strategy S_4 is applied with an input I_1.

Conclusions

The present article is one kind of survey of research work undertaken by me and my research scholars in transcendental fuzzy measures of information and discrimination. These measures have very interesting properties and have found wide applications in Science and Engineering problems.

The above mentioned work was extended further by Mishra *et al.* [24] by defining and characterized 'Weighted Trigonometric and Hyperbolic Fuzzy Information Measures'. They also studied their application in optimization theory.

Prakash *et al.* [25] also studied new measures of weighted fuzzy entropy and their applications for the study of maximum weighted fuzzy entropy principle. Recently, transcendental fuzzy measures of information and discrimination have invited attention of researchers and hence, a lot of work in this area of research is in progress.

Acknowledgement

The author is extremely thankful to the reviewers for their comments and suggestions for improvement of the article.

References

1. Shannon CE (1948) A mathematical theory of communication. Bell Syst Tech J 27: 379-423 & 623-659.
2. LA Zadeh (1965) Fuzzy sets. Inform Contr 8: 338-53.

3. Singpurwalla ND, Booker JM (2004) Membership function and probability measure of fuzzy sets. J Amer Stat Assoc 99: 867-77.

4. Zadeh LA (2002) Towards a precision-based theory of probabilistic reasoning with imprecise probabilities. J Stat Infer 105: 233-64.

5. Zadeh LA (2005) Towards a generalized theory of uncertainty (GTU) – an outline. Inform Sci 172: 1-40.

6. Yager RR (2002) Uncertainty representation using fuzzy measures. IEEE Trans Syst Man Cybernet Part B 32: 13-20.

7. Klir GJ (2004) Generalized information theory: aims, results and open problems. Reliab Eng Syst Safety 85: 21-38.

8. Nanda AK, Paul P (2006) some results on generalized residual entropy. Information Science 172: 27-47.

9. DeLuca A, Termini S (1972) A definition of non-probabilistic entropy in setting of fuzzy set theory. Inform Contr 20: 301-12.

10. Bhandari D, Pal NR (1993) Some new information measures of fuzzy sets. Information Sciences 67: 204-28.

11. Hooda DS (2004) On generalized measures of fuzzy entropy. Math slovaka 54: 315-25.

12. Hooda DS, Bajaj RK (2008) On generalized R-norm measures of fuzzy information. Journal of Applied Mathematics, Statistics and Information 4: 199-212.

13. Mishra AR, Hooda DS, Jain D (2015) On exponential fuzzy measures of information and discrimination. International Journal of Computer Applications 119: 01-07.

14. Hooda DS, Jain D (2009) Sub additive measures of fuzzy information. Journal of Reliability and Statistical Studies 2: 39-52.

15. Sharma BD, Taneja IJ (1977) Three generalized additive measures of entropy. Elec Inform Kybern 13: 419-33.

16. Monte S, Causo L, Gil P, Bertoluzza C (2012) Divergence measures between two fuzzu sets. International Journal of Appropriate Reasoning 30: 91-105.

17. Hooda DS, Mishra AR (2015) On Trigonometric Fuzzy Information Measures. ARPN journal of Science and Technolog 5: 145-52.

18. Prakash O, Gandhi CP (2010) Applications of trigonometric measures of fuzzy entropy to geometry. World Academy 0f Science, Engineering and Technology 37: 468-71.

19. Bhatia PK, Singh S (2013) A new measure of fuzzy directed divergence and its application in image segmentation. International Journal of intelligent system and applications 4: 81-9.

20. Bhandari D, Pal NR, Majumder DD (1992) Fuzzy divergence, probability measure of fuzzy events and image thresholding. Pattern Recognition Letters 1: 857-67.

21. Poletti E, Zappeli F, Ruggeri A, Grisan E (2012) A review of Thresholding strategies applied to human chromosome segmentation. Computer Methods Programs Biomedicine 108: 679-88.

22. Fan S, Yang S, He P, Nie H (2011) Infrared electric image thresholding using two dimensional fuzzy entropy. Energy Procedia 12: 411-9.

23. Ghosh M, Das D, Chakraborty C, Roy AK (2010) Automated leukocyte recognition using fuzzy divergence. Micron 41: 840-6.

24. AR Mishra, Hooda DS, Jain D (2016) Weighted trigonometric and hyperbolic fuzzy information measures and their optimization principles. International Journal of Computer and Mathematical Sciences 3: 62-8.

25. Prakash O, Sharma PK, Mahajan R (2008) New measures of weighted fuzzy entropy and their applications for the study of maximum weighted fuzzy entropy principle. Information Sciences 178: 2389-95.